Geometrical objects and figures in practical, pure, and applied geometry

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§1. Introduction

THE ROLE OF DIAGRAMS IN MATHEMATICS, and in particular in Euclid's pure geometry has been addressed in many works.[1] Here, we are interested in a very particular subject: that of the conception of geometrical object or geometrical figure and their relation, when this is the case. We adopt the term geometrical figure to identify simple drawings like the drawing of a segment, a circle, a square, a rectangle, or a triangle. The term diagram is reserved for composite drawings in which we combine several figures. One example is the famous diagram in the proposition 1 of book 1 of the *Elements* (I.1). This diagram can be seen as composed of several segment figures and two circle figures (see figure 1 below).[2]

In this work, we will consider three kinds of (planar) geometries: the practical geometry of ancient Mesopotamia, the pure geometry of ancient Greece, and ancient Greek applied geometry. The objective of the work is to make precise the differences, in these geometries, in the conception of geometrical object or geometrical figure, and their relation. For that purpose, we adopt Panza's approach to geometrical objects and diagrams in Euclid's *Elements* (Panza 2012). Panza's approach enables to make precise what we might regard as the intuitive grasp of the differences regarding objects and figures that we find when comparing practical, pure, and applied geometry.

In section 2 we address the sortal conception of geometrical objects and their representation by geometrical figures in Euclid's *Elements*, focusing in particular in the attributes of objects resulting from their conceptualization and those inherited from the figures. Section 3 addresses geometrical figures in the

1 For reviews see, e.g., Shin, Lemon, and Mumma (2018); Giardino (2017).

2 The term 'geometrical figure' corresponds to Panza's 'elementary diagram' (see Panza 2012, p. 72).

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practical geometry of ancient Mesopotamia and the lack of a conceptualization of geometrical objects. Section 4 addresses the applied geometry of Euclid's *Optics*, focusing on new layers of representation arising in applied geometry.

§ 2. On geometrical objects and figures in the planar geometry of the *Elements*

For our purpose, it is not necessary to have a clear notion of geometrical object. For example, we could think of a geometrical object, adopting a Platonic view, as a purely ideal one or, adopting an Aristotelian view, as an abstraction from a concrete one (Panza 2012, 57, footnote 5). The relevant features we will be interested in are the relation of a geometrical object to a concept and its relation to a geometrical figure.

Panza takes geometrical concepts to be sortal concepts.[3] Accordingly, "the assertion that some objects fall or do not fall under them is meaningful" (Panza 2011, p. 46). How do we have, e.g., a sortal concept of circle? This concept, like any other, is characterized by what we might call the application conditions and the identity conditions of the object —in this case, the geometrical circle— that falls under it (see, e.g., Panza and Sereni 2013, pp. 100–1; Hale 2013, p. 176, footnote 29). In this way: "The former are necessary and sufficient conditions for an object to fall under this concept [and] the latter are necessary and sufficient conditions for the objects that fall under this concept to be distinct from each other" (Panza 2011, p. 46).

Panza develops his views on geometrical concepts in relation to the first six books of Euclid's *Elements* and Euclid's *Data*. According to Panza, in relation to equilateral triangles:

> Definitions I.19–20 of the *Elements* fix the application conditions of the concept of equilateral triangle. They do it by stating that equilateral triangles are rectilinear figures contained by three equal segments. (I understand this statement in this way: an object falls under the concept of equilateral triangle if and only if it is a rectilinear figure contained by three equal segments). These same definitions do not provide, however, identity conditions for the objects that possibly fall under this concept. Moreover, they provide neither a warrant that some objects actually fall under it nor appropriate norms for obtaining some such objects. (Panza 2011, p. 47)

3 For a brief account of the notion of sortal concept see, e.g., Mackie (2006, pp. 118–21).

The application conditions of the concept of circle are determined in the definitions 15 and 16 of book 1 of the *Elements*: "A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another; and the point is called the center of the circle" (Euclid 1956, pp. 153–4).[4] However, we still need the identity conditions of the concept of circle. Also, if we do not consider the object to be given before the concept, we need "a warrant that some objects fall under the relevant concept, and/or some norms for obtaining some such objects" (Panza 2011, p. 47) (e.g., we might have the sortal concept of flying saucer but not a warrant that some 'object' falls under it; see Panza 2011, p. 46, footnote 8). According to Panza, in relation to equilateral triangles, "the warrant that some objects actually fall under the concept of equilateral triangle is provided by the solution of proposition I.1 of the *Elements*" (Panza 2011, p. 48). In the case of the circle, this warrant is given by the third postulate of the *Elements*: "Let the following be postulated: to draw a straight line from any point to any point. To produce a finite straight line continuously in a straight line. To describe a circle with any center and distance" (Euclid 1956, p. 154).[5] Notice that here we are not simply saying that we are allowed to draw a circle as a figure, e.g., using a compass. Interpreting 'circle' in terms of a geometrical object, we would be drawing a figure —a circle figure— that has a particular representation relation with the geometrical object that we call circle, but which is distinct from the drawn figure. As we can see in the application conditions for the concept of circle under which the object must fall, the concept of circle (and its instantiation as a geometrical object) is that of an object for which all radii are exactly equal (this cannot be the case for a drawn figure; in this case we can only have a measured equality which has always associated an uncertainty; see section 3).

Regarding the identity conditions for the circle (as a geometrical object), we must consider the special kind of representation made by diagrams in the *Elements*; i.e., the relation between the circle as a figure or diagram and the circle as a geometrical object. According to Panza, geometrical figures or diagrams have a two–fold role in the *Elements*. On one hand, the identity conditions of the concept under which falls an object "are provided by the identity conditions of the diagram that represents [the object]" (Panza 2012, p. 57). For example, in I.1 we draw two contour–closed curves, each of these figures provides the identity conditions enabling to identify and distinguish the

[4] Taking a 'cue' from Netz (1999, p. 94), we do not present the definitions or postulates numbered and separated in lines.

[5] We reproduce as a continuous piece of prose the first three postulates of the *Elements*.

geometrical objects each one represents (see figure 1). That is, when we identify —or provide an individual reference to— each circle figure as a particular one (the circle ACE and the circle BCD) and at the same time distinguish them, we are also identifying and distinguishing the corresponding geometrical circles ACE and BCD. On the other hand, the geometrical object, in this case the geometrical circle, "inherits some properties and relations from these diagrams" (Panza 2012, p. 58).[6] In the case under consideration, the continuity of the geometrical circle and the fact that the circles cross at a point C is inherited from the concrete continuity and crossing of the circle figures. Accordingly: "Continuity is a diagrammatic property of [geometrical circles] to the effect that they are continuous just because the concrete lines that represent them are taken to be so" (Panza 2012, p. 76); and: "These circles meet in a point C, which, so to say, pops up, because of the physical properties of the lines that have been drawn or imagined. The point is, indeed, represented by the intersection of these lines" (Panza 2011, p. 51).

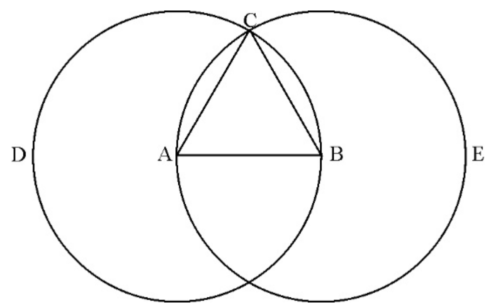

Figure 1. The diagram in I.1 of the *Elements*.

There is a subtle point regarding this last part. It is related to an attribute of geometrical objects resulting from the application conditions of the geometrical concepts in the *Elements*, which Ferreirós calls "the 'Euclidean' idealizations" (Ferreirós 2016, p. 123). This can be noticed already in the first three definitions of the *Elements*. According to definitions 1 to 3, "a point is that which has no part. A line is breadthless length. The extremities of a line are points" (Euclid 1956, p. 153).[7] According to Ferreirós:

[6] Following Panza, we will use sometimes the term 'attribute' when referring either to properties or relations (Panza 2012, p. 75).

[7] The 'Euclidean' idealizations imply that «a point is characterized as a non–measurable entity, as it has no parts that can measure it […] a line is measurable with respect to its length, while it is non–measurable with respect to its breadth» (Harari 2003, p. 18).

> The first definitions indeed suggest a *way of reading diagrams*, a perspective for seeing or conceiving what is implied by a diagram, and what is not. And this way of reading is not at all evident, especially if one previously knows only practical geometry. For the definitions and the reading that comes with them lead the practitioner to certain crucial idealizations. More importantly, the definitions suggest certain forms of response (and of indifference) to some aspects of the diagram: thus, the crossing of two drawn lines will be a (very small) planar region, but we are taught to disregard this and consider in the argumentation that one and only one point has thus been determined. (Ferreirós 2016, p. 144)[8]

In terms of Panza's approach, it is still the case that the crossing of geometrical circles is inherited from the concrete continuity and crossing of the circle figures. However, that this crossing is made at a point results not from the diagram but directly from the idealization imprinted to the geometrical concepts. We have, so to speak, a dialog between the geometrical object and its corresponding geometrical figure (and diagram where it is embedded). The geometrical object inherits properties and relations from the drawn diagrams but is reinterpreted in terms of the attributes of the geometrical sortal concepts.[9]

Another important attribute of geometrical objects in the context of the *Elements* derives from their implicit treatment in several parts of the propositions in terms of magnitudes (which are explicitly addressed in books 5 and 6). According to Mueller: "The most appropriate interpretation of magnitudes in the *Elements* involves construing them as abstractions from geometrical objects which leave out of account all properties of those objects except quantity: i.e., length for lines, area for plane figures, volume for solids, size, however characterized, for angles" (Mueller 1981, p. 121).

As we can see in the application conditions of the sortal concept of circle as expressed in definitions 15 and 16, a circle is such that all radii are exactly equal. Here, the attribute of radii (geometrical segments) as magnitudes is essential. In fact, throughout the book 1 we can notice that segments can be taken to be equal (e.g., in I.1), or one being greater than or less than another (e.g., in I.2). That is, an attribute of geometrical objects as magnitudes is the possibility of establishing "the relation of equality and order" (Panza 2012, p.

8 In a footnote, Ferreirós mentions that «indifference and responsiveness to features of representations is a topic that I have seen Manders elaborate on in an unpublished conference» (Ferreirós 2016, p. 144).

9 In this work, we only address planar geometry, as Panza does. Our figures and objects are planar. The Euclidean plane can be seen as another instance of idealization, e.g., of a practically planar surface where drawings were made (possibly a dusted surface, a wax tablet, or others; see, e.g., Netz 1999, pp. 14–6).

77).

Addressing circles as magnitudes together with the idealization that the circumference or radii of the circle are breadthless lengths is in fact at the crux of the sortal concept of circle. As mentioned by Ferreirós: "The object [of geometry] is the diagram as perceived and interpreted, in light of the idealizations and exact conditions superimposed on it by the text, by Euclid's theoretical framework. Hence, the figure studied by the geometer is the ideal type [the geometrical object], not the empirical token [the diagram]" (Ferreirós 2016, p. 139).

No circle figure can ever have radii drawn exactly equal or have a circumference whose intersection with another curve is a point. This is particularly clear in the simple case of the exact equality of geometrical segments. This does not mean "that the concrete lines that represent them are so" (Panza 2012, p. 77). Two drawn lines cannot be exactly equal; the most we can achieve is to consider them as practically equal when considering the uncertainty in the length measurements to be negligible (see section 3). We see that the geometrical object called circle cannot be confused with a drawing called circle. The sortal concept of circle in the *Elements* cannot correspond to an actual circle (e.g., a figure or a shape).

We find Panza's framework, who was conceived for Euclidean planar geometry, a valuable analytical tool to interpret other forms of geometry and compared between them. In the following sections we will address (aspects of) the geometry of ancient Mesopotamia and ancient Greek applied geometry, in relation to ancient Greek pure geometry, in the light of Panza's approach.

§ 3. On geometrical figures in the geometry of ancient Mesopotamia

Several tablets from ancient Mesopotamia bear witness of the use of some sort of compass to draw the geometrical figure we call circle; we also find a clear depiction of what we call the center of the circle. Could this mean that the conceptualization of circle in ancient Mesopotamia was, e.g., like that of Ptolemy that conceptualized the circle in terms of a diameter rotating about its center? (Robson 2004, p. 20)

Circles like other geometrical figures were conceptualized in terms of their boundaries. According to Robson "a circle was the shape contained within an equidistant circumference" (Robson 2004, p. 20). As it is, the name for circumference and for circle was the same, and, in fact, corresponded to that of the circumference. It meant "thing that curves" (Robson 2004, p. 20). The length of the circumference like the length of other horizontal lines is measurable in terms of a 'sanctioned' length unit or subunit; for example, in the Old Babylonian period an important unit of length was the *ninda* or *rod*, corresponding approximately to 6 meters (Cooper 2013, p. 403).[10] To measure the length of a circumference considered, e.g., as the concrete boundary of a circular oven (Cooper 2013, p. 404), one uses a rope with a defined length.[11] The length of the circumference is then given in terms of what we might call a measure number (Gonçalves 2015, p. 14).

What about the area of a circle? The notion of area, for which is used the word corresponding to a field plot in a slightly technical sense (Høyrup 2002, pp. 35–6), was not conceived directly in terms of the measure number of a unit area. That is, the 'surface' of, e.g., a field is not thought as being covered with unit squares (each one corresponding to the same measure number of one base unit in the adopted area metrological system). The numerical value of the area of, e.g., a field or planar shape, was determined from the measure numbers of the boundary (Damerow 2016, p. 101). In this way, e.g., for quadrilateral field plots (or figures), surveyors calculated the area of the field using the so–called surveyors' formula in which the area is given by the multiplication of the mean values of the two sets of two opposite sides; i.e., as $(A + B)/2 \times (C+ D)/2$, in which A, B, and C, D, are opposite sides of a quadrilateral (Damerow 2016, pp. 106–7). The notion of the area of geometrical figures derives from this notion from practical geometry (Damerow 2016, pp. 115–7). Like for any other figure, the area of a circle is determined by the length of its boundary; it is given by the square of the length of the circumference divided by 12 (Robson 2004, p. 18).

Let us look into ancient Mesopotamian mathematical problems in the light of their practical geometry. Let us consider a mathematical problem like the one in tablet MS 3051. Here, we have an equilateral triangle inscribed in a circle, whose circumference is drawn using a compass (see figure 2).[12] Some conventions must be taken into account to understand the problem. One is that in Old Babylonian mathematical texts it is implicit that the length of a circumference is 1 00 *ninda* (60 *ninda*). The area of a circle with a circumference of 60 *ninda* is, as given by the formula for the circle's area, 5 00 *sar* (300 *sar*) (Friberg 2007, p. 207).

10 For details on the (changing) ancient Mesopotamian metrological systems see, e.g., Robson (2008, pp. 291–5).

11 The use of ropes as instruments to measure length is attested, e.g., in Robson (2008, pp. 120–2) and Baker (2011, pp. 296–7).

12 For a reproduction of a picture of tablet MS 3051 see, e.g., Friberg (2007, p. 488).

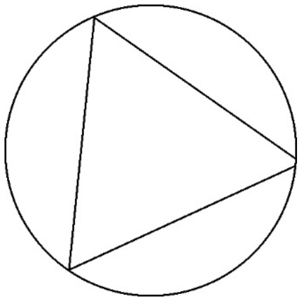

Figure 2. Reproduction of the drawing in tablet MS 3051.

How should we consider the circle drawn in tablet MS 3051? Is it a representation of a geometrical object? In fact, what notion of geometrical object might we consider in this context? There is a sense in which the circle in MS 3051 is a representation. The tablet is small; it has a couple of centimeters of width and length (Friberg 2007, p. 190). Evidently, if by convention the circumference is taken to have a length of 60 *ninda*, the drawing is representing a larger figure. But if we had actually drawn this figure, e.g., with a peg at the center and using a rope with 10 *ninda* (60 meters) to which we attach, e.g., a small stick (to enable to draw the circle in the ground), what is this that we drew? A representation of a geometrical object? In the context of a practical geometry, a circle, like other planar geometrical figures, has concrete instantiations, e.g., as the boundary of a field. When we draw or mark this boundary, we are literally given shape to a portion of land by creating a frontier that separates it from the outside. The shape determines the inside —our figure. We can certainly consider some notions of shapes previous to the conceptualization proper. A notion of circular shape is manifested in early Mesopotamian visual culture in sketchy drawings or in the form given to some artifacts (see, e.g., Robson 2008, pp. 45–6; Frankfort 1970, pp. 17–37). By the time of the Old Babylonian period, we already assist to a conceptualization of the geometrical figure (Robson 2004). This has several components. That the circle is conceptualized is made explicit by naming it. As mentioned, the conceptualization is made in terms of a boundary —the circumference— giving shape to the figure that we name after it. The circle can be related to a concrete instantiation, like a circular oven that is determined by a circumference, but it can also be related to a small drawing on a surface like the clay tablet. The drawings can be very sketchy but also quite precise (see, e.g., Friberg 2007, p. 190). This precision in the drawing is obtained by using a particular instrument to draw the circle —a compass (Høyrup 2002, p. 105; Friberg 2007, p. 207). To the circle, we have associated several numbers, the most important of which is the length of the circumference, which can be measured. The area, the diameter, and the radius can be calculated from the measured length of the circumference (however, we might consider measuring directly the lengths of the diameter and the radius).

As we have seen, to the geometrical figure drawn in a tablet is attributed the measure number of 60 *ninda*. For our purpose, we might as well consider the actual measure number of the drawn circumference. From our contemporary perspective, an important thing is that in relation to experimentation a measure number cannot be exact. There is always an uncertainty associated with any measurement. To say that the length of the circumference is 60 *ninda* is a shortcut to say that the measurement gave the result 60 ± the uncertainty, in which the uncertainty depends on the precision of the measuring instrument (Hughes and Hase 2010, pp. 2–6). A contemporary counterpart of an Old Babylonian circle would be a drawing made with precision instruments that enabled the uncertainty to be very small in relation to the measured value of the length of the circumference.

Can we consider a practically constructed circle a representation of a geometrical object? To address this issue, we need to determine what kind of concept of circle we have in Old Babylonian mathematics. We do not have application conditions in the form of a definition —we only have a name. Naming the circle is only meaningful, e.g., when considered in relation to a drawn figure. It cannot be considered as an application condition on its own. We might consider the practical procedure of drawing a circle —using a compass— as our application condition. However, this 'application condition' refers directly to the drawn figure, not to anything else. Taking the construction of a circle using a specific instrument as its application conditions would mean that we already have a warrant that the figure falls under the concept. This can be seen the other way around. The possibility of drawing a circle with a compass —its warrant— is at the same time its application condition. Considering this 'charitable' view on application conditions, what about identity conditions? In a way similar to that of the *Elements*, we might consider that a circle figure gives a sort of identity conditions of the concept of circle. However, this does not imply, by itself, that there is a geometrical object that falls under the concept. These identity conditions are related to the drawing itself. Lacking application conditions that, so to speak, go beyond the drawn figure, as such, the drawing might refer only to itself and not to some object that falls under the concept.

The point is that we do not have a sortal concept of circle. If we try to make the application conditions —the necessary and sufficient conditions for an object to fall under the concept (Panza 2011, p. 46)— as redundant due to the priority given to the drawing using a compass taken as a warrant that the figure falls under the concept —in a way being this a warrant for having a concept by drawing figures in this very specific way—, the most we might arrive at is at a make–believe sortal concept of circle figure. We would have a sort of warrant–application condition in terms of a drawing of a contour–closed curve using a compass which we name 'thing that curves'.

As we have seen, according to Panza, in the *Elements*, the identity conditions of the object that falls under the concept "are provided by the identity conditions of the diagram that represents [the object]" (Panza 2012, p. 57). Also, the geometrical object, in this case the geometrical circle, "inherits some properties and relations from these diagrams" (Panza 2012, p. 58). In the case of the drawn circle in Old Babylonian geometry, we can certainly talk about the identity conditions of the figure. There are also properties and relations determined by the diagrams. For example, in MS 3501 we have the drawing of a triangle inscribed in a circle. In the *Elements*, an attribute like "lying inside" (Panza 2012, p. 76), is a relation inherited by the geometrical objects from the diagram. However, in Old Babylonian geometry it refers to the figures in the diagram directly. This is so because in ancient Mesopotamian geometry there is no sortal concept of geometrical circle (or equilateral triangle) in relation to which we might consider that the geometrical object inherits attributes from the figure or diagram.

The limitation, when compared with the *Elements*, of the practical geometrical conception of circle in Old Babylonian geometry is not the inexistence of a written definition of circle; it is the lack of idealizations and exactness attributes as given, e.g., in the definitions of line and circle in the *Elements*. This means that there is no abstract circle taken to be exact in ancient Mesopotamia for which the drawing might be taken to be a representation. As we have mentioned, a drawing can be seen as a representation of another drawing but nothing else. In fact, the most 'exact' circle we have in ancient Mesopotamia is a circle figure drawn with a compass. More than a 'warrant' the act of drawing a circle (or the implicit postulation that this can be done), corresponds to the articulation of the concept. The concept of circle is established by drawing a circumference, e.g., using a compass, naming it, and having implicit the conceptualization of planar figures in terms of the inside of a boundary that can be measured. That ancient Mesopotamian geometry is a practical one implies that all the lengths associated with the drawing are measurable. This means that we simply cannot have an attribute like 'all radii are equal' in the case of the conceptualization of circle (if it had been written out). The most we could achieve is to say that as measured all radii have the same measure number in the adopted metrological system. From our contemporary perspective we might even say that the uncertainty is negligible; i.e., that for practical purposes the radii are equal. But this refers to a very precisely drawn figure itself, not to the geometrical object it might be taken to represent.

§ 4. Applying the geometry of the *Elements*: geometrical objects and figures in applied geometry

For this section, it will suffice to look closely into the concept of segment. In ancient Mesopotamia, we have measurable segments. Accordingly, in mathematical problems, "the old Babylonian calculator represents his unknown numbers by his standard instruments —measurable line segments" (Høyrup 2002, p. 57). We can imagine drawing a line figure using a straightedge and measuring its length —as a measure number— using a 'sanctioned' rod, which corresponds to a metrological unit and possibly having marks for its subunits.[13] When segments are in a particular relation between them as part of the boundary of certain figures they can be named accordingly. In the case of a square, its name derives from 'to confront'; the square "understood as a confrontation of equal sides and thus a square frame, *is* the side and *has* an area" (Høyrup 2002, p. 25). The side of a square in some texts is referred to as "the equalside" (Høyrup 2002, p. 26). In the case of rectangles, these are determined by the 'long side' and the 'front' (making reference to actual field plots in which the 'front' is the narrow side parallel to the irrigation channel; Høyrup 2002, p. 34). We must bear in mind that in the case of the square the 'equalsides' are not exactly equal sides, the most we can say is that they are practically equal. In the same sense, that one side in the rectangle is greater than or less than another is a practical relation of order. We might say that we establish a practical relation of order between measure numbers. This cannot be confused with "the relation of equality and order" (Panza 2012, p. 77) existing between segments as geometrical objects. Like in the case of the circle, it does not make sense to consider a geometrical segment that falls under a sortal concept of segment. We only have the figure (and some notion of segment).

[13] While there is no known example of a measuring rod with marking of subunits in ancient Mesopotamia, there are examples from ancient Egypt. The so–called cubit–rods «are staffs that indicate the length of one cubit and may include its subdivisions» (Imhausen 2016, p. 168).

In the case of the *Elements*, we can consider, following Panza, the definitions related to segments (or straight lines as they are called there) as giving the application conditions of the sortal concept of segment. The identity conditions and several attributes are given by segment figures that represent the geometrical segments. Again, it is an attribute of a geometrical segments as magnitudes the exactness of their length and the relation of order between geometrical segments. This entails that two segments can be exactly equal one to another or one being greater than or less than another. Finally, like in the case of circles, the postulates warrant the possibility of having geometrical segments. Again, when drawing a straight line using, e.g., a straightedge we are drawing a segment figure that represents a geometrical segment.

Let us now consider the geometrical segment and its representation as a drawn straight line in relation to Euclid's *Optics*. The underlying idea of this work is that the eyes emit a 'visual fire' described within Euclid's geometry of rays as geometrical segments called 'visual rays'. The visual perception arises with the incidence of the visual rays on the surface of objects and the observer is taken to be aware of the directions of these visual rays (Darrigol 2012, p. 8).

As mentioned by Ferreirós, "when applying geometry, e.g., to study geometrical optics or mechanics, new layers of representation enter the picture" (Ferreirós 2016, p. 124). It had already been noticed by Toulmin that the development of geometrical optics corresponds to "the application of new modes of representation" (Toulmin 1953, p. 43); it consists in a "geometrical method of representing optical phenomena" (Toulmin 1953, p. 26). These views apply, *mutatis mutandis*, to the case of Euclid's geometry of rays, which can be seen, even if just regarding the adoption of Euclid's geometry, as a precursor of geometrical optics (Darrigol 2012, pp. 10–1).

Some of the assumptions made at the beginning of Euclid's *Optics* are: "The straight lines drawn from the eye diverge to embrace the magnitudes seen. The figure contained by a set of visual rays is a cone of which the apex is in the eye and the base at the limits of the magnitudes seen. Those magnitudes are seen upon which visual rays fall, and those magnitudes are not seen upon which visual rays do not fall" (cited in Darrigol 2012, p. 9). For our purpose it will suffice to consider the first proposition of the *Optics* (see figure 3):

Prop. 1: No observed magnitude is seen simultaneously as a whole.

Call AD the observed magnitude, and B the eye from which the visual rays BA, BG, BK, BD fall. Since the visual rays diverge, they do not fall on the magnitude AD in a contiguous manner; so that there are intervals of this magnitude on which the visual rays do not fall.

> Consequently, the entire magnitude is not seen simultaneously. However, as the visual rays move rapidly, it is as if we saw [the entire magnitude] simultaneously. (cited in Darrigol 2012, p. 10)

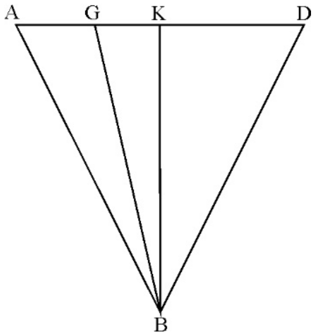

Figure 3. The diagram in proposition 1 of the *Optics*.

Here, we are in the realm of applied geometry. As we can see in the assumptions of Euclid's *Optics*, the geometrical segment (the straight line) is taken to represent what we might call a physical entity —an optical phenomenon that we might call the visual fire. In this way, having a new layer of representation we might call the geometrical segment a visual ray making reference to this new representational role. The visual fire emitted by the eye is described solely as a geometrical object —the visual ray (i.e. as a geometrical segment). Also, like in the case of visual rays, already in the assumptions, concrete extensions are described as magnitudes. This is made clear in proposition 1, where a segment AD corresponds to the observed magnitude. The eye, which has a central role in Euclid's *Optics* as the emitter of the physical entity described geometrically as visual rays, is represented in the assumptions as the apex of what we might call the visual cone. This, when addressed in terms of planar geometry, as in proposition 1, reduces to a point: the point B in the figure represents the eye. Regarding the visual rays, these correspond in proposition 1 to the geometrical segments BA, BG, BK, and BD.

What is the relation of the drawn figure to the physical entities represented by geometrical objects? The particular relation of representation between geometrical figures and their corresponding geometrical object is the same as that addressed in section 2. However, now we take the geometrical objects to represent physical phenomena or entities (a concrete extension, the eye, the visual fire). This means that in applied geometry, the geometrical figure has a double representational role, but the new role is indirect. The geometrical figure (or diagram) represents the geometrical object, and this represents a physical entity. In this way, the geometrical figure represents the physical entity, via the geometrical object represented by the figure.[14]

One more aspect must be noted regarding applied geometry when compared to pure geometry. It is not simply that geometrical objects are given a representational character. This is made in the context of physical assumptions regarding what is being represented by the geometrical objects. For example, the first assumption, at the beginning of the *Optics*, that the straight lines drawn from the eye diverge to embrace the magnitudes seen, is a geometrical rendering of a physical assumption. The eye is here already the apex of a geometrical cone; the 'seen' object is a geometrical object (a segment, a rectangle, the arc of a circle, a sphere, or others; see Burton 1945), and the visual fire emitted by the eyes is a visual ray —a geometrical segment. These initial assumptions are complemented by what we might call further physical assumptions made in the propositions, like in the case of proposition 1 where we find the idea of "a fast scanning of the objects by the visual rays" (Darrigol 2012, p. 10). As with the case of the initial assumptions, this further physical assumption is presented already in terms of an assumption regarding geometrical objects: in proposition 1 visual rays are taken to "move rapidly" (Darrigol 2012, p. 10). This can be seen as a new 'layer' of representation of the optical phenomena. We do not have only the first new layer of representation (i.e., the eye, the concrete objects, and visual fire, as geometrical objects) that bears on points and geometrical segments on their own. We also have a representational role given to the diagram as a whole, which has implicit the physical assumption of the scanning of the concrete objects by the visual fire emitted by the eye. This new layer of representation implies taking the segments BA, BG, BK, and BD as 'moving rapidly', which means, e.g., the eye emitting initially BA and emitting successively BG, BK, and BD. The diagram would be a static image of a dynamical situation. In this way, this further representational role implies a particular reading of the diagram, which, in fact, is made explicit in the text of proposition 1. We can regard this further 'layer' of representation as implemented directly in the diagram, not in the individual geometrical segments whose representations —the segment figures— form the diagram. This is so because the complete physical situation of the eye emitting successively the visual fire in different directions so as to 'cover' the surface of an object, is described by the diagram and not by the individual segment figures or point figure that represent geometrical segments or a geometrical point (which represent the visual fire, a concrete extension, or the eye). There is no

14 Notice that here we are not making the case for a new interpretation of the role of geometrical figures that would be different from Panza's. The new layers of representation are built, so to speak, on top of Panza's 'original' layer. We still follow Panza's view here.

'whole' geometrical object that is represented by the drawn diagram. However, the diagram has this representational role because of the representational role that was given to the geometrical objects represented in the diagram.

§ 5. Conclusions

In the practical geometry of ancient Mesopotamia, the conception of segments, circles, or others, is such that we cannot consider having a conception of geometrical objects. A geometrical figure does not represent a geometrical object. For example, we can have a concept of circle figure in terms of a general conceptualization of planar figures as determined by its boundaries —in this case, the circumference—, that is specified implicitly by the stipulation of its drawing with a compass and a name that makes reference to it as a 'thing that curves'. The drawn circumference has a length that is measurable in relation to an adopted metrological system for lengths using, e.g., a small rope.

The situation with the pure geometry of the *Elements* is distinct. We have a sortal concept of segment, circle, and others. In the attributes of these conceptions, we find idealizations like lines having no breadth and the notion of exact length. This implies, e.g., that we can compare the length of two (geometrical) circumferences in terms of a relation of equality and order. That is, the length of two (geometrical) circumferences can be exactly equal or one is greater than or less than the other. However, the circle figure is not the geometrical circle instantiating the concept of circle. Contrary to the case of the practical geometry of ancient Mesopotamia, the circle is not the figure, the figure is a representation of the (geometrical) circle. According to Panza, the identity conditions of circles is given by the circle figures (or diagrams they are part of). In this way, we identify and distinguish two circles referred to in I.1 through the circle figures of the diagram in I.1. Also, the continuity of the circles as geometrical objects is inherited by them from the concrete continuity of the drawings. Another attribute inherited from the diagram is the crossing of the circles. However, that they cross at a point does not follow directly from the diagram; it follows from the diagram complemented by the attribute that objects have due to their idealization as breadthless.

When applying Euclid's pure geometry in Euclid's *Optics*, we encounter new layers of representation. These operate in the geometrical objects represented by drawn segments or points or directly in the diagrams. The geometrical segments can represent concrete extensions or objects and, more importantly, optical phenomena. As visual rays, they represent a visual fire emitted by the eye which permits the visual perception. The eye itself is simply represented as a geometrical point. When considering the representation of a concrete extension, the visual fire, or the eye, as segment figures or a point figure within a diagram, this diagram provides a further layer of representation of physical assumptions. In the case of the diagram of proposition 1, the diagram represents the visual 'scanning' of an object by the successive emission of visual rays by the eye.

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Geometrical objects and figures in practical, pure, and applied geometry

The purpose of this work is to address what notion of geometrical object and geometrical figure we have in different kinds of geometry: practical, pure, and applied. Also, we address the relation between geometrical objects and figures when this is possible, which is the case of pure and applied geometry. In practical geometry it turns out that there is no conception of geometrical object.

Keywords: Sortal Concepts · Diagrams · Elements · Optics · Mesopotamia.

Objetos geométricos y figuras en la geometría práctica, pura, y aplicada

El propósito de este trabajo es abordar qué noción de objeto geométrico y figura geométrica tenemos en diferentes tipos de geometría: práctica, pura y aplicada. Además, abordamos la relación entre objetos geométricos y figuras cuando esto es posible, como es el caso de la geometría pura y aplicada. En la geometría práctica resulta que no existe una concepción del objeto geométrico.

Palabras Clave: Concepto Sortal · Diagramas · Elementos · Óptica · Mesopotamia.

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