The (reasonable) effectiveness of mathematics in empirical science

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ABSTRACT

I discuss here the pragmatic problem in the philosophy of mathematics, that is, the applicability of mathematics, particularly in empirical science, in its many variants. My point of depart is that all sciences are formal, descriptions of formal–structural properties instantiated in their domain of interest regardless of their material specificity. It is, then, possible and methodologically justified as far as science is concerned to substitute scientific domains proper by whatever domains —mathematical domains in particular— whose formal structures bear relevant formal similarities with them. I also discuss the consequences to the ontology of mathematics and empirical science of this structuralist approach.

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The (reasonable) effectiveness of mathematics in empirical science

Jairo José da Silva

SINCE EUGENE WIGNER'S FAMOUS PAPER (Wigner 1960) much has been said about the uses of mathematics in science, particularly as a heuristic tool. Wigner thought this was a "gift" we "neither understand nor deserve"; others, as the philosopher Mark Steiner (Steiner, 1989, 95, 98), went as far as claiming that the fact that mathematics can be used in science as an instrument of discovery puts a problem for the belief that man has no *special* place in the natural order of things.

I want to argue here, first, that the scientific usefulness of mathematics is mysterious only from an empiricist point of view, that the mystery utterly disappears in an idealist perspective, and, second, that the transcendental–idealist approach to mathematical structuralism that I sketch here can explain to full satisfaction the four main uses of mathematics in empirical science, the representational, the organizational, the predictive and the heuristic. I also show how to deal, from this perspective, with ontological issues in mathematics (the existence of mathematical objects and Benacerraf's dilemma and the arguments of indispensability advanced in favor of mathematical Platonism) and science.

1. Let me begin with a fundamental logical fact, essential for the discussion here. Any theory, in no matter which language and how carefully designed, which has an interpretation has *always more than one*. In the optimal case of *categorical* theories, all interpretations are isomorphic. In this case, and in this only, one can say that the theory *univocally* characterizes *something*, namely, *the same identical ideal form* or *structure* instantiated in each of its interpretations. Non–categorical theories can be seen either as incomplete characterizations of ideal structures that are, nonetheless, *assumed* to be *completely* determined *in themselves* (i.e. independently of the theories) or as complete characterizations of families of structures. No theory can by itself singularize a *materially determined* domain of interpretation.

For the terms of a language to have a *material* content beyond the *formal* content that they have as terms of a grammatically structured language, or, in other words, for them to *denote* something, to *grasp* a material content, something beyond language is required, namely, an *intentional act* on the part of the language user that associates content to symbols: objects of a material ontological type to object–names, relations among these objects to relation–names, etc. Denoting is an *intentional* act that cannot, at its *most basic* level, be linguistically expressed. Only *after* a material content is given to the basic symbols of a language, this language can be used to singularize objects and situations in the domain of

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reference in question via definitions and descriptions. No interpretation of a theory can be singled out independently of a reference being determined by non–linguistic means.[1]

A few definitions are in order. A domain is *materially determined* or, equivalently, a *material domain* if its objects belong to a *proper* subdomain of the most general ontological domain, that of *all* objects. For example, the domain of *physical* objects is a material domain, as is that of *ideal* objects. A domain is only *formally determined* or a *formal domain* if its objects do not belong to any *particular* ontological domain; i.e., if they are taken as nothing beyond objects in the most general sense. Formal domains can be given either by formal (i.e. non–interpreted) theories as their correlates or by *formally abstracting* the form of material domains. Formal abstraction is essentially *a logical operation* by which entities of particular ontological types are taken as entities of the most general ontological types to which they can belong (this operation has a correlate in the operation of *des–interpretation* in which symbols are devoid of any particular interpretation).

Given a materially determined *structured domain*, i.e. a domain of objects of a particular type where structuring relations are defined, the *abstract form* or (*formal*) *structure* of *this* domain is the domain *itself*, but ignoring the ontological specificity of its elements, which are "seen" as nothing but unspecified objects. By "being seen" I mean that their *specific* material properties are not taken into account. Of course, to be seen as unspecified objects is a *logical–intentional*, not psychological operation. In other words, the objects of the given domain, despite belonging to a particular ontological category, are intended only as objects simpliciter; the determinations they have as the specific type of objects they are are not taken into consideration. An *abstract* structure is, in other words, a formal structured domain whose elements and structuring relations are taken as nothing beyond instances of the formal categories of *Object* and *Relation* respectively, their material nature being abstracted out, that is, ignored (which, of course, does *not* mean that they do not exist).

The structure of a *given* structured material domain is an abstract aspect of *this* domain and cannot subsist without it (objects are *abstract* when their existence requires the existence of other objects on which they ontologically depend, like the color of the colored object). But one can, by identifying the abstract structures of all *isomorphic* domains, which are *equal* but *not identical*, posit an *ideal* structure that is instantiated in all those domains. Although abstract objects can be *real*, such as the color of a physical body, which is located at the same chunk of space–time where the object whose color it is is, or still the abstract structures of structured systems of real objects, *ideal* structures are *not real*, even if they have real instantiations, and, then, are not located in space–time.

Ideal structures can also be determined as *objective correlates of non–interpreted categorical theories*, i.e. as the structures these theories posit descriptively. Theories in general, and categorical theories in particular, are intentionally loaded systems of propositions which purport to describe *something*. In case of *interpreted* theories, these things are their *standard* interpretations, which in general pre–exist the theories that describe them. In case of *non–*

[1] In more technical terms, the *interpretation function* is not definable in the language.

interpreted or *pure* theories, however, these things do not pre–exist their theories and are *posited* by them. The objective correlates of pure theories are materially indeterminate structured domains, determined only as to form but not content. *Categorical* non–interpreted theories, in particular, posit uniquely characterized structures identically instantiable in all their material interpretations.

We must keep this in mind: no theory, pure or interpreted, can independently of accompanying intentional acts singularize a materially determined domain. Empirical theories are no exception. No matter what we think empirical theories are theories of, transcendent reality or only our perception of it, the fact remains that empirical sciences can only reach the formal–structural skeleton of their domains.[2] By itself, language only captures the formal, the material requires interpreting language, which cannot be accomplished within the language one is interpreting.

Therefore, a *material* theory is essentially a theory of the *ideal* structure of a materially determined structured system of things, often the *standard* interpretation attached to the theory. But any material theory can be reinterpreted as the theory of any domain that shares with the standard interpretation all the formal properties that the theory captures and expresses. Categorical theories are the only theories that encapsulate enough formal properties of the standard interpretation to single out its ideal structure, instantiated in *all* its interpretations, i.e. all structured domains isomorphic to the standard interpretation (it may, however, if it is not *deductively complete*, fail to determine *deductively* all the formal properties of this structure that are expressible in the language of the theory). In this sense, all theories, materially determined or not, are formal. I will call the intended interpretation of an interpreted theory its *standard semantics*.

2. It is a relevant epistemological fact that the theoretical development of a theory does *not* depend essentially on its standard semantics, even when the theory is not syntactically complete (when it is, everything that is true in the domain of the theory is deducible in the theory). One can develop the theory of a domain by investigating any domain that has relevant formal similarities with the domain of the theory, since theories can only capture formal aspects of their domains and, therefore, cannot discriminate between domains that share relevant formal properties. In case the theory is categorical, *any* of its interpretations can serve as a field of investigation as far as the theory is concerned. An interesting example of this methodological strategy in mathematics is analytic geometry, in which geometry is done algebraically by solving numerical equations instead of carrying out geometrical constructions. In general, in case of only partial formal similarity, two domains will share some, but maybe not all formal–structural properties.

The applicability of mathematics to daily life, mathematics itself or the empirical science, depends solely on this fact. *Mathematics is the theory of ideal structures in general and its*

[2] For methodological reasons, structures “extracted” from perceptual “reality” are often idealized and extended mathematically, as we will see below.

applicability and practical effectivity depends on the availability of mathematical structures that are instantiable in domains of practical or scientific interest or domains that have relevant formal similarities with those that interest us in practical affairs and theoretical science.

3. Let us consider the applicability of mathematics in empirical science in more details. Our first approach to the empirical world is perceptual, and science must be built on this basis. Even if theoretical science usually goes well beyond it, perceptual reality remains the focus of interest and the highest court where scientific theories are tested.

But perceptions are not made of sensations *only*. Perceptual reality is a *construct* based on sensorial data but not *reducible* to them. There is more to perception than what meets the senses. What we perceive is the result of both built–in psychophysical systems that organize sensorial inputs into percepts and a certain amount of intentional action that *interpret* what one is perceiving. Perceiving event A causally related to event B, for example, goes beyond simply having a bunch of sensorial impressions followed in time by another bunch of sensations. Perception is *organized sensation* with *meaning*, it has both *hyletic* (sensorial) matter and *categorial* form, which is, at least to some extent, a contribution of the perceiving subject.

So, *perceptual reality* is *already* a construct. But not yet a mathematical construct; to become one, further intentional action, such as abstraction and idealization, are required. These actions "extract" the abstract structure out of perceptual reality, further idealizing it into a mathematical manifold proper, *mathematical–perceptual reality* to give it a name. In mathematical physics, the mathematical exactification of the abstract structure of perceptual reality —i.e. mathematical–perceptual reality— usually stands for reality *itself*; it is *real reality*, so goes the view, that perception can only approximate. *Anything in perceptual experience that cannot be given a formal correspondent in mathematical–perceptual reality is not of interest to physics*. It is discarded as a subjective experience with no objective relevance. Only *form* is objective. The subjective sensation of color or heat, for example, only find a way into objective science by being made to correspond to objective entities that can be mathematized, that is, represented mathematically, frequency of electromagnetic radiation and temperature in our examples, both representable in the continuum of real numbers.

Then, a *correspondence* is established between perceptual reality and mathematical structures so that the latter *represent* the former.[3] For mathematical physics, however, mathematical–perceptual reality *is* reality.[4] In *Crisis* (Husserl 1954) Husserl criticizes this

3 Husserl refers explicitly to a correspondence between the system of perceptions (*objectified sensations*) and its mathematical representative: "[...] the specifically sensible qualities […] that we experience on bodies given to intuition *are intimately connected according to a rule*, in a particular manner, to the *forms* that belong to them according to their essence" (Husserl 1954, §9c). Still, according to him, the perceptual world has "*its double in the realm of forms*, in a way that *any change in the manifold of contents* [*the perceptual world, my note*] *has a causally induced copy in the sphere of forms*" (ibid., §9h).

4 Husserl is explicit about this: "[…] the general *hypothesis* (*my emphasis*) according to which empirical Nature is experienced as an approximation of the mathematically ideal Nature" (Husserl

emphatically; mathematical–perceptual reality, he points out, is *not* reality, only a methodological devise for indirectly investigating the *formal* aspects of perceptual reality (or transcendent reality, depending on one's metaphysics) by mathematical means.

Mathematical–perceptual reality is only the materially empty formal mold, the *structure* of perception in *idealized mathematical form*. It can only represent perceptual reality (or transcendent reality that supposedly "causes" it) by means of a *semantics*, either that which was "abstracted out" in the mathematization of perceptual experience or any other consistent with it. *Mathematical–perceptual* reality can be dissociated from its *standard* semantics, the perceptual reality at its basis, and be given *another* semantics. The same is true of the mathematically enriched extensions of mathematical–perceptual reality (this alone accounts for the *heuristic* applicability of mathematics in science, as we will see).

Theories of mathematical physics are *not* tested by being confronted with perceptual reality *directly*, but with already intentionally elaborated mathematical–perceptual reality. Hence, as Husserl points out in *Crisis*, *transcendental presuppositions* that go into the intentional constitution of mathematical–perceptual reality are not directly tested experimentally.

The intentional constitution of perceptual reality, even before its idealization into mathematical–perceptual reality, involves a series of presuppositions that are *not* hypotheses or *empirical* presuppositions that can be put to test, but real *transcendental* constitutive presuppositions. The more relevant ones, on which the very *logic* of reasoning about reality depends, are that reality is a consistent, ontologically complete, objectively given, cognitively accessible domain of being. The world is such that *objects* and *facts* of the world preserve their identity in the flux of time, facts do not rule out one another (consistency), no situation (*representation* of a fact) that is *possible* in the world is in itself indeterminate as to its factuality (ontological completeness), the world exists out there (objective giveness) and is ideally, i.e. in principle, knowable (cognitive accessibility). The perceptual world, however, is not only the *actually* perceived world but the ideally stable maximally consistent system of things and facts *in principle* perceivable.[5] Mathematical–perceptual reality is its abstract structure mathematically idealized.

4. But mathematical–perceptual reality is only a first and still incomplete (mathematical) sketch of what the mathematical science of nature calls *empirical reality*.[6] Empirical reality contains mathematical–perceptual reality but extends further than the perceivable, even the

1954, appendix IV to §12).

5 A situation of the world is in principle perceivable if represents the world in accordance with the meaning attached to the categories by which the world is structured.

6 I call *empirical reality* the objective correlate of the mathematical sciences of nature. Empirical reality is a mathematical manifold, not because nature is *intrinsically* mathematical, but because it is *made* so for methodological purposes. (Mis)taking mathematical surrogates of perceptual reality for "true" reality, which perception can only imperfectly access, is, for Husserl, the characteristic feature of the modern mathematical empirical science.

only in principle, only imperfectly perceivable, into a domain of purely theoretical constructs with no representational role (for example, the wave–function of quantum mechanics). Theories of mathematical–perceptual reality, the so–called *phenomenological theories*, often only organize perceptual experience in terms of *ad hoc* principles and rules (for example, the early quantum mechanics of Planck, Bohr and Einstein). Full–blown mathematized science further extends mathematical–perceptual reality mathematically to have access to more refined *mathematical* methods to deal with it. It is a methodology.

This depends on the fact that, by being a mathematical manifold, mathematical–perceptual reality can be immersed into other mathematical manifolds by having further elements, structure, or both, adjoined to it. The mathematical–physicist is to some extent free to mathematically "improve" mathematical–perceptual reality, *provided observable consequences of the resulting theories are testable and turn out to be true*. There is *no* a priori restriction on which mathematics and how much mathematics can be used in the theoretical development of physical theories, for beyond the level of the mathematical *representation* of perceptual reality (mathematical–perceptual reality) *mathematics does not necessarily a have a representational role in science.*[7]

Some questions, however, are always on the scientist's mind: 1) are assertions derived in physical theories with the assistance of the mathematical machinery available therein that can be translated back *via the standard semantics* into situations that are *in principle* observable observed, that is, are they *true*? 2) Can objectual, conceptual and relational terms of physical theories that are *not* a priori representational and assertions involving them derived therein be given an interpretation in perceptual reality by consistently extending and enriching the *standard* semantics?

5. What has been said suggests that mathematics has essentially the following uses in science:

a) *Representational*, by expressing in idealized form abstract aspects of perceptual reality. In this role mathematics is a *language* that *represents* perceptual reality via a fixed interpretation. Perceptual reality has an abstract structure that mathematics expresses in idealized form.

b) *Organizational*, by providing contexts of immersion of mathematical–perceptual reality. Mathematics used for organizational purposes may or may not represent something in perceptual reality. In this role mathematics is a *tool* that "constructs and delivers structures", a provider of contexts where mathematical inferences can be carried out. A good example is linear algebra in matrix mechanics; only some elements of this mathematical formalism represent something experienceable, the rest has only an organizational role *interna corporis*.

c) *Predictive*, by allowing the derivation of theoretical assertions that can be given an

[7] As Husserl insists, *empirical* reality, that is, reality as conceived by the mathematical science of nature, is only a construct devised for methodological reasons. Among them, to provide better ways to *predict* future perceptions.

observational content, either in standard or extended, non–standard semantics. If previsions turn out to be *true*, the mathematics involved in their derivation has a bigger chance of having a representational character, but not necessarily, for *semantically meaningless mathematics can be used for deriving semantically meaningful assertions that happen to be true*.

d) *Heuristic*, by providing *calculi* that admit *non–standard* semantics where assertions that are not meaningful in the standard semantics become meaningful. If the extended semantics turns out to correspond to experience, that is, if it provides a *new* standard semantics, one is tempted to say that mathematics was "unreasonably effective" in showing how reality is. But this is not correct. The modified semantics is not *required* by the mathematical formalism, for no formalisms can determine a semantics. Moreover, there may exist *more than one* semantics that turns meaninglessness into meaningfulness (example, the theoretical "prediction" of positrons, which can be given different interpretations). And, finally, the modified semantics may not at all correspond to reality. Mathematics *alone* has no heuristic virtues; at best the mathematical formalism can, by its rules of symbolic manipulation, allow the emergency of *formal* regularities that have no meaning in the standard semantics, but that can be given one by modifying this semantics. How, mathematics is silent about.

6. As said before, mathematics is the science of formal structures (forms, patterns, whatever you want to call them). The applicability of mathematics in science requires, first, that a *mathematical* structure be attached to perceptual experience. This, as we have seen, requires intentional action. And second, that a sufficiently rich plethora of structures be available where mathematical–perceptual reality, the mathematical idealization of abstract aspects of perceptual reality, can be *immersed* for methodological reasons.

Purely mathematical extensions of mathematical–perceptual reality need not correspond to anything effectively perceived or potentially perceivable, and even less to anything metaphysically *real*, although it is not ruled out a priori that it can. Their utility rests solely on the fact that their theories may offer a more convenient context for the formal–structural investigation of mathematical–perceptual reality.

The mathematical investigation of contexts of immersion of mathematical–perceptual reality may disclose *formal* facts that can be given an interpretation in either standard semantics or convenient extensions of it. In the first case mathematics plays a predictive role, in the second, a heuristic one. Predictions can be either confirmed or disconfirmed perceptually; in case they are disconfirmed the *entire* theory responsible for the prediction is under threat.

Heuristic suggestions, on their turn, are only vindicated when there actually are extensions of standard semantics, that is, extensions of "standard" perceptual reality that offer a better context of interpretation for the mathematical extensions of the original mathematical–perceptual reality. In this case, one can say, the utility of mathematics in science rests on its ability to disclose *formal* aspects of a hidden reality *before* we *experience* this reality directly. But, it is important to notice, whether in fact there is such a reality or what it is if it exists are things mathematics is completely silent about. This undermines the *strong* claim that mathematics has, *by itself*, a heuristic role in science. Mathematics heuristic virtues can only bloom irrigated by the semantic creativity of the scientist.

7. The structuralist approach to mathematics that I have sketched here can account, as I think to have shown, for the many ways mathematics appears in empirical science; in particular, as a heuristic tool. I want now to address two ontological questions; the first, whether there are mathematical objects other than empty structures and, the second, whether the indispensability of mathematics in modern empirical science does indeed require the existence of mathematical objects. More precisely, I want to address briefly the so–called Benacerraf's dilemma and a version of an argument known in the literature as the *indispensability argument*. Let us begin with Benacerraf.

The fact that mathematical theories are formal (i.e. they are structural descriptions of either instantiated or pure structures) does not necessarily require that mathematical objects do not exist, for if one accepted this conclusion for mathematics one would, out of coherence, had to accept it also for no matter which science, including empirical science, for *all* theories, including empirical theories are, as already stressed, formal in this sense.

Mathematical objects exist, even though their theories are not particularly concerned with them from a *material*, only from a *formal* perspective. Let us consider one example, material (or contentual) arithmetic, whose domain is that of numbers in the proper sense of the term, i.e. objects falling under the *concept* of number, the *schema* —to use Kantian terminology— of the category of quantity.

A number —or, better, a natural number— is an *ideal form*, namely, the *quantitative form* of a class of equivalent collections of whatever objects. Two collections are equivalent when they are equinumerous, i.e. when there is a 1–1 correspondence among them. Number or quantitative form is *that* which equinumerous collections have in common. Numbers can be ordered as to their magnitude, n being smaller than or equal to m when there is an injective application, not necessarily onto, of *any* collection numbered by n into *any* collection numbered by m. One can also operate with numbers, m being the *sum* of n and k when m is the number of the *disjoint union* of *any* two collections numbered by n and k, and so on. The domain of arithmetic is the domain of numbers structured by these operations and relations.

Contentual arithmetic can be divested of its domain, that is, formally abstracted, and reinterpreted as the theory of a different domain, in particular any domain that is isomorphic to the domain of numbers. One can then say that contentual arithmetic is, but at the same time that it is not a theory of numbers. As a contentual theory it is, but it can only express, as *any* other theory, that which is most formal about its domain, namely, that which can be reinterpreted in other material domains that are formally similar to its original, intended domain, that of numbers proper. In this sense, arithmetic is not exclusively *about* numbers, but anything that *behaves like* numbers. Or better still, *contentual* (material) arithmetic is the *formal* theory of numbers *qua* numbers, whereas *formal*, non–interpreted arithmetic, is the formal theory of number–like entities in general.

This ambiguity has induced some philosophers into believing that there is nothing more to number than being places in a structure, i.e. numbers are numbers only because they behave as numbers, thus effacing completely the difference between contentual and formal arithmetic. Paul Benacerraf, for example, has a famous argument purporting to show this: since numbers can be interpreted in infinitely many equivalent ways as sets or, for that matter, objects of any type, numbers are *not* objects, but materially empty entities that only exist in a system. Numbers, he thinks, have no non–structural characterizing properties, nothing that distinguishes them from other number–like entities. Numbers only exist in a system, depending on other numbers to be characterized and singularized.

Against Benacerraf, I claim that numbers are singular objects of a type, namely, idealized abstract quantitative forms belonging to a domain circumscribed by a concept that can be intuitively accessed. Numbers can also be intuitively accessed individually, if not effectively, at least as a matter of principle. The *number* 1,117, for example, is a well–determined object materially (but not formally) different of other objects that may occupy its place in the ideal number system. 1,117 is not merely the name of a place in an empty structure, but of a *number*, in principle, if not actually accessible to intuition by abstracting and ideating the quantitative form of a collection of objects that can in principle be presented to consciousness.[8]

Moreover, although equinumerous collections instantiate the *same* number, numbers are *not* collections of equinumerous classes, *although they can be thus represented from a purely formal perspective*. Nor can numbers be reduced to any collection they number. In short, the fact that one can *represent* numbers set–theoretically does not imply that numbers *are* sets. Numbers are what they are, ideal abstract forms instantiable in quantitatively determined collections of objects as their quantitative form.

Hard core structuralists believe that numbers cannot be told apart from other number–like entities. This is true, of course, if one remains confined to arithmetic, for any theory can be reinterpreted salva veritate as the theory of a domain materially (but not formally) different from its intended domain. But there are *non–arithmetic* ways of characterizing numbers as the specific objects they are, and *intuitive* ways of accessing numbers by abstracting the quantitative form out of given collections of objects. So, numbers can be told apart from objects of other materially determined interpretations of arithmetic and be directly presented to consciousness independently of arithmetical properties. Numbers have material and formal properties, the former tell them apart from any other objects, other number–like entities in particular, the latter are those they have in common with number–like objects in general. The

8 *Ideation* is the intentional experience that posits ideal objects, *the* number 5, for example, over and above the many 5's instantiated as abstract aspects of collections of 5 objects.

most fundamental material property of numbers is that they are *quantitative forms*, a property, however, inexpressible in arithmetic. Contentual arithmetic only captures the formal properties of the numerical domain as a system of objects, operations and relations, i.e. *as a number-like domain*. The fact that only the formal properties of numbers are of interest to mathematics does not imply that they do not have material properties *tout court*.

Mathematics is a *formal* science, in the precise sense that mathematical theories are formal, and the true objects of mathematics are formal structures, instantiable in principle in any material context whatsoever. But mathematics often finds it useful to posit material domains conceptually (for example, that of numbers proper) with the *double* purpose of instantiating a structure in it *and* accessing this structure via the concept that circumscribes the domain. The material nature of objects is, from a mathematical perspective, irrelevant, although it often provides a means of accessing the structure they instantiate.

Although a theory may be born out of an intuitive grasp of the concept delimiting its material domain, such as, for example, contentual arithmetic, it does not have to be tied to this domain for its development. Any formally similar domain —in particular, isomorphic ones— will do. If the other domain is more intuitively accessible than the original domain, or more easily investigable, then substituting the domain of a theory by another will constitute a methodological advantage. This strategy is widely popular in mathematics (e.g. analytic geometry) and accounts for many of its applications. In general, domains are not strictly isomorphic but share enough formal properties for the theory of one to be useful for investigating the (formal) properties of the other. Even *empirical* domain, conveniently abstracted and idealized, can be *replaced* by mathematical domains in the mathematical sciences of nature.

As just said, to posit a mathematical structure, it is sometimes convenient to posit a domain of objects as the objectual correlate (the extension) of some concept or other and consider its structure idealiter. One can in this case access the structure of the domain via the concept that delimits it. Material structured domain serve, via their ruling concepts, as means of epistemic access to the ideal structure they instantiate. For example, one can access the ideal ω-structure, the structure of the domain of natural numbers, by accessing the concept of natural number via conceptual intuition. The properties of the *ideal* ω-structure, however, do not depend on the material nature of the domain that instantiates it, numbers proper or number-like entities.

By quantifying conveniently idealized physical magnitudes, such as time, space, temperature, volume, pressure, and a plethora of others, quantitative relations among magnitudes can be expressed in terms of mathematical formulae that can be used to predict future behavior of physical systems. With a proviso, that the original magnitudes remain the domain of interpretation of mathematical formulae. Similarly, one can interpret geometrical constructions in terms of numerical operations and use algebraic manipulations, not geometrical constructions, to solve geometrical problems.

The fact that mathematical theories can only capture and express formal–structural properties of their domains, or any others formally similar to them, and that the applicability of mathematics in some domain depends on the formal–structural properties of this domain being in some relevant way similar to those expressed by some mathematical theory, makes the material nature of mathematical objects, such as number, irrelevant, either for theoretical or practical purposes. In mathematics, matter does not matter.

One could say that mathematics is not *really* about mathematical objects but about the formal structures they instantiate. Although *there are* mathematical objects, mathematics is interested in them only as supports of structure, and "invented" structures are as good in theory and in applications as those instantiated in pre–existing structured domains.

In short, the practical utility of mathematics does not imply that *there are objectively existing* and *objectively independent* domains of *mathematical* objects that happen to be structured as their theories determine. Mathematical theories are in some sense useful fictions. Their utility depends on the existence of relevant logical relations between structures of theoretical interest, often empirical structures, and mathematical structures that may not correspond to anything existing. Mathematics need not, *necessarily*, to be *true* in the sense of describing an *independent* realm of abstract objects to be useful in science (although it may). The scientific usefulness or even indispensability of mathematics has *no* ontological consequences. This, I believe, undermine indispensability arguments purporting to prove mathematical Platonism from the essential role mathematics plays in science.

8. The mathematization of natural science raises important ontological questions concerning reality. Here are some: what should we understand by *empirical reality* (or, for short, *nature*), that which we can *in principle experience* with our senses, that is, *perceptual reality*, some *transcendent reality* that "causes" perceptions or, still, the perceptually inaccessible *mathematical manifolds* that *replace* perceptual reality with which science, insofar as it is a mathematical science, is immediately concerned? Can transcendent reality be *itself* a mathematical manifold, which, supposedly, we can at best access only imperfectly through the senses? If not, what is mathematics doing in science?

The mathematical theory of empirical reality is, obviously, not *directly* concerned with perceptual reality, only with mathematical "models" of it. Perceptual reality must be intentionally lapidated, elaborated, before becoming a suitable subject of scientific consideration. The question is whether these mathematical models are reality itself or only useful methodological devises to deal with it. Are mathematical models of perceptual reality methodological tools or aspects of reality itself as disclosed to *reason*?

Mathematization is a complex operation and it would be enlightening to consider it again, with some concrete examples this time.

The first moment in the constitution of mathematical models of reality is *subrogation*, i.e. the substitution of objects of perception by mathematical representatives or surrogates. For example, *ideal* geometrically well–determined abstract spatial extensions "physically" characterized by numerically expressible *objective* properties such as mass, volume, temperature, electrical conductivity, etc., in place of *real* physical bodies and the *immediate sensorial impressions* they produce. Color and like properties, for example, deemed *subjective*, are usually dismissed as unworthy of playing a role in *objective* science if they cannot be *correlated* to *objective* quantifiable properties in terms of which they can be "explained" away. The color of a body, for instance, "reducible" to the frequency of the electromagnetic radiation emitted by the body that reaches the eyes.

Mathematization requires a considerable amount of intentional action on the raw material of perception, the most evident being *abstraction* and *idealization*, which I sometimes refer to as cutting and polishing, respectively. For example, if one is interested in the movement of bodies in space (kinematics) from a mathematical perspective, that which the body is made of, its matter or substance, is irrelevant; the scientist can ignore or *abstract* it out. If changes in the state of movement and their *causes* are of concern (dynamics), one can still abstract from the substance of the body, provided that we attach to it a property (which can or cannot depend on the state of movement) —its mass— that intermediates between causes (forces) and effect (acceleration). In both cases, if the dimensions and geometrical properties of the geometric extension to which the body was reduced is irrelevant, it can be idealized as a massive *point* occupying the "center of mass" of the body, itself an ideal, geometrical point that does not belong in real space, only in the geometrical space where the "body" —or better, its geometrical ghost —now lives.

Abstraction consists in "separating" an aspect of the body —the cutting—, for example, its shape, from the rest of it. Of course, not in reality or "in the mind", but for the sake of theoretical considerations. Abstraction is a logical operation. *Idealization* —the polishing—, on its turn, consists in a process of exactification; for example, the *real* shape of the body, which is an object of perception, not geometry, occupying a place in real space, being substituted by an exact geometrical form supposed to be materialized in the body but belonging in mathematical, not real space.

There is a considerable number of *presuppositions* involved in the process. For example, that the mass of bodies varies *continuously* and can be *precisely* expressed in terms of a standard mass *as real numbers*. Needless to say, this presupposition does not express a perceptual fact; it is simply taken for granted until a force majeure, a chance in theoretical frame, for example, forces science to give it up.[9] By *quantifying* the concept of mass; i.e. by considering it only in terms of quantitative relations, the content of the concept —*what* it is, its *qualitative* aspects— is pushed out of scientific considerations, the *quantitative* aspects only remaining.

Maybe the most important goal of science is the search of regularities in nature that allow the correct prediction of its future behavior. In the mathematical science of nature, this is done *indirectly*. A certain correspondence is established between perceptual reality and

9 Science, for instance, had to give up the presupposition that the energy of a system can in general take any value in a continuum of possible values.

"mathematical models" so that we can investigate *formal aspects* of perceptual reality by investigating their models by appropriate mathematical methods. As already observed, this process happens against a background of presuppositions. *Mathematical models are mathematically exactified quasi–isomorphic copies of selected abstract aspects of perceptual reality*. At the most basic level, they are idealizations of the perceptually discernible formal structure of perceptual reality (I call it mathematical–perceptual reality), but can be further elaborated mathematically beyond even the possibility of perception.

Since only the *formal* aspects of perceptual reality find a way into scientific reasoning, it is *methodologically* advisable to ignore the matter of perception and consider only its form, which can be mathematically idealized and submitted to mathematical scrutiny. One can now look for *mathematical regularities* in the mathematical manifolds representing in ideal form abstract aspects of perception and go back to perceptual reality by the inverse operations of subrogation.

One can credit Galileo, among others, for discovering this very efficient way of investigating empirical reality. Suppose, for instance, that we are, as he was, interested on how the space travelled by a free–falling body relates to the time that takes it to complete the fall. We can simply measure both magnitudes and try to find out a mathematical expression correlating both sets of measurements. This requires the operation of mathematization described above. But notice, we would not be *describing* perceptual regularities, but regularities in a *mathematical substitute* of perceptual reality. Once we find that distance travelled depends quadratically on time ($s = ct^2$, where c is a numerical constant), we can use this formula to *predict* the time it would take *any* free–falling *physical body* to cover *any* distance, *not only those we have actually measured*.[10] But to go from the mathematical model to perceptual reality symbols must be given a material content, i.e. re–interpreted in perception. For this we usually resort to the standard semantics, which remains in place.

As I have notice before, to put the prediction to empirical test requires more than simply checking it against raw perception. The prediction is not tested in perceptual experience, but in its surrogate, the mathematical model. Upon measuring one of the variables, say, time, a *real* number must be selected to represent it. There are literally infinitely many possible choices within a fuzzily determined interval, the choice is arbitrary within this interval. This is often *misinterpreted* as an approximation to the *truly real* value of the variable, as if the mathematical were the real and perception only an attempt, doomed *in principle* to failure, to capture it.

Now, once a number is *chosen* to represent time, it can be substituted in the formula and the corresponding value of distance be determined by *algebraic manipulations*. The number determined via the formula can now be compared with the number *chosen* to represent the value obtained in a *direct* measurement; the formula passes the test if these numbers are "sufficiently close". One does not compare the formula *directly* with perception, *but with the*

10 Of course, this generalization involves a presupposition concerning regularity in the behavior of nature.

mathematical substitutes we put in place of perception; theoretical predictions are not validated in perception, but in mathematical surrogates of it.

But once the first mathematical draft of perception is in place, there is no limit to how much it can be mathematically enriched. Instead of wanting simply to establish a correlation between time and space in the free falling of heavy bodies, for instance, one could ask *why* they fall. Aristotle had a quick answer: because they are eager to get to their natural place, the center of the earth in case of heavy bodies (i.e. bodies that fall), which coincided with the center of the universe. From this, he concluded, quite reasonably but wrongly, that heavier bodies would fall faster than lighter ones. Newton, who only considered inertial movements as natural, reasoned differently; for him, acceleration demands a cause, a *force* acting upon it. Of course, no such "force" is *directly* detectable in perception; it is a theoretical contribution that can only be given a content in terms of mathematical relations: force is directly proportional to acceleration, the (inertial) mass of the body being the constant of proportionality. The second law of Newton is in fact an *operational mathematical definition* of force. Once perceptual reality has been reduced to a mathematical manifold, there is no reason why *purely mathematical entities* like forces could not be introduced therein. As to the precise nature of forces, that which corresponds to them in a possible *direct* perception, Newton preferred not to advance hypotheses: *hypotheses non fingo*, he said.

Examples of such theoretical constructs are plenty, fields, potentials, state–functions, wave function, etc. etc. Clearly, they are not elements of perceptual reality, nor are they *necessarily* required for an adequate treatment of the phenomena at hand; the scientist may choose other approaches. For example, instead of massive bodies exerting forces on each other in a Euclidean space (an Euclidean physical space being also, obviously, an idealization) they may move by inertia in a non–Euclidean space whose structure depends on the distribution of the bodies in space. The history of physics is the history of the *dematerialization of nature*, the stuff of perception either put aside or given a mathematical correspondent. Only the form of perception survives, which can be idealized in mathematical form and mathematically enriched *arbitrarily*.

Here, things can get very interesting. How do purely mathematical theoretical constructs relate to perception? There are two obvious possibilities; first, they do *not* correspond to anything at all in perception, they have only a role *interna corporis* in the organization of the theory: by allowing, for example, the derivation of theoretical predictions that are in principle verifiable in the standard semantics (an example would be the wave function of quantum mechanics). Second, they can *themselves* be given a *material* content, *either in the standard semantics or in another that extends it*.

Perceptual reality provides the standard context of interpretation of the mathematical models that take its place in mathematized natural science; the former is to the latter as matter is to (idealized) form. However, after mathematical idealizations of abstract aspects of perceptual reality, i.e. mathematical–perceptual reality, are, as they can be, mathematically enriched, theoretical entities can appear that do not or cannot, even in principle, correspond to anything in perceptual experience. The question then presses itself: can they be *made* to correspond to *something* in empirical reality? The answer to this question, however, is beyond the power of mathematics and is left for the ingenuity and creativity of the scientist. At best mathematics can display interesting *formal possibilities* whose *material reality* is completely undetermined.

Mathematical theories of empirical reality are tested by verifying whether the *purely formal* mathematical predictions of the theory can be translated back, via the *standard* semantics, into *true* assertion in perceptual reality, whenever these predictions can in fact be so translated. A second, higher–level test, is by verifying whether *perceptual* reality can *itself* be extended to fit mathematical "predictions" that are *not* translatable into standard semantics. If *perceivable* consequences of the theory (in the standard semantics) are not *perceived*, the *whole* theory is under threat if one cannot identify and fix what is wrong with it. As Weyl claimed, a scientific theory stands or falls as a whole. On the other hand, if all the theoretical predictions of the theory are actually verified, if only provisionally, this does *not* imply that all theoretical constructs of the theory have material reality. Scientific theories exist where "imaginary" entities only play a role *interna corporis*.

Now, if "imaginary entities" can be given a material content in an *extension* of *standard* semantics, one can say that mathematics played a *heuristic* role in science, but only at the purely formal level. The *actual* semantic extension falls completely outside the mathematical domain; by itself, mathematics is heuristically barren.

In face of all that, how can we answer our original question: what are we entitled to call (empirical) reality, perceptual reality or its mathematical idealization?

One may, of course, introduce a *tertium*, a transcend reality that somehow manifests itself in perception but is not intrinsically mathematical. However, if we can have an undistorted *perception of* transcendent reality, we can simply identify both realities, perceptual and transcendent. If not, a transcendent reality has no place in science. What if perception does not simply *mirror* transcendent reality, but instead distorts it beyond repair? In fact, we *know* this to be the case, perception bears to a considerable extent the stamp of the perceiver. As already emphasized, perceptual reality is a *construct*; it is how we *happen* to perceive a transcendent reality that may, in itself, be beyond perceptual reach (if it makes sense to say that transcendent reality has an intrinsic way of being expressible in our conceptual systems, even though not capable of manifesting itself perceptually). Perceptual reality may be only a *perspective* of something out there that is heavily conditioned by the nature of our perceptual system, but it is all that we have.

On the other hand, as already sufficiently emphasized, mathematical models of perceptual reality, mathematical–perceptual reality or their mathematical extensions, are only methodological devises that are certainly useful for the investigation of formal aspects of perceptual reality but are *not* reality. In short, and here is my answer, nature is not a mathematical manifold, although it can be conveniently studied by mathematical methods insofar as science is only concerned with its formal–structural aspects. Now, to the extent that a transcendent reality beyond perceptual reach has no place in science and we cannot take for reality what is only a methodological devise, the conclusion imposes itself that empirical reality is perceptual reality and its *abstract* structure the *true* object of science. In the words of Heisenberg (Heisenberg 1959): “the science of nature does not deal with nature itself, but with nature as man considers and describes it” and those of Weyl (Weyl 1952, p. 26) talking about physical geometry: “… our conceptual theories enable us to grasp only one aspect of the nature of space, that which moreover, is most formal and superficial.”

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